

# Models of decoherence with negative dephasing rate

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We determine the total state dynamics of a dephasing open quantum system using the standard environment of harmonic oscillators. Of particular interest are random unitary approaches to the same reduced dynamics and system-environment correlations in the full model. Concentrating on a model with an at times negative dephasing rate, the issue of “non-Markovianity” will also be addressed with the emphasis on information obtained from the dynamics of the total state of system and environment: making use of criteria that allow us to distinguish between classically correlated and entangled total states, we employ a simple measure for the correlations emerging from the increase of the two local entropies, and relate it the nature of the correlations.

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## I. INTRODUCTION

In a realistic description of open quantum system dynamics one has to account for influences of the external environment [1–3]. Despite the unitary time-evolution of the total state of system plus environment, the dynamics of the system itself will in general be non-unitary. Growing correlations between the system of interest and its surroundings lead to a decay of the initially present coherences. This line of thought is at the heart of decoherence theory and is put forward to explain the appearance of classical properties in quantum systems [4–8]. Decoherence in particular is of relevance for quantum technologies trying to make use of the vast computational potential forecast to applied quantum information processing [9].

In the regime of weak system-environment coupling and short environmental correlation times the dynamics of an open quantum system may be described in terms of the Born-Markov approximation. The corresponding Markov master equation then has a generator in Lindblad form [10, 11]. The time-evolution of the system depends on its present state only. Often, however, such an approximation is not justified. Then, memory effects—typically incorporated by means of integrals over the past [12, 13]—start to play an essential role. Yet, it is known that for arbitrary bath correlation functions exact time-local master equations can be derived [2]. An important example is given by the exact master equation for a damped harmonic oscillator bilinearly coupled to a bath of harmonic oscillators [14, 15]. Here, we focus on a dephasing open quantum system using the standard harmonic oscillator environment.

In more recent developments, the question of how to distinguish Markovian from non-Markovian dynamics from a local (system) perspective was addressed. The analysis has been based both on a single snapshot of the dynamics [16] and on the full time evolution of the open

quantum system within a certain time interval [17, 18]. In the latter approach, memory effects associated with non-Markovian dynamics are expected to cause temporary increase in the distinguishability of states (in terms of trace distance, e.g. [17]). Under Markovian dynamics, on the other hand, the decay of distinguishability will be monotonic throughout.

The role of system-environment correlations in open quantum system dynamics has raised some notable interest lately. In the context of quantum discord [19], for example, total states with no quantum correlation (zero discord) were shown to be the most general class of initial states allowing for completely positive reduced dynamics [20, 21]. Another interesting point is the relation between open quantum system decoherence and system-environment entanglement. Here, it is possible that the system decoheres completely without becoming entangled with its environment at all [22, 23].

Talking about the dynamics of open quantum systems, the flow of quantum information from system to environment and vice versa is an often employed, and certainly intuitive picture. However, since there is a lack of a quantitative description, e.g. in terms of a continuity equation, such a picture should be used with caution. Care has also to be taken with respect to the need for a proper environment. At this point it should be noted that for single-qubit decoherence the dynamics may always be described in terms of stochastic fluctuations of external fields, i.e., the dynamics has a random unitary representation [24, 25]. The dynamics may thus be modeled without invoking a quantum environment at all. Higher dimensions are needed to see proper quantum dynamics [26].

In order to shed light on the role of “quantumness” in open system dynamics in a non-trivial (non-Lindblad) regime, we here investigate single-qubit dephasing [23, 27, 28], by means of two different approaches: we use a proper quantum model and also show how to find a random unitary representation of the same reduced dynamics. In the quantum case, we employ a useful representation of the total state of system plus environment discussed in [23]. This enables us to study the dynamics of both system and environment purity, as well as the

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dynamics of system-environment correlations, which we also relate to non-Markovianity.

## II. QUANTUM DECOHERENCE MODEL: REDUCED AND FULL DYNAMICS

Continuing our earlier work on the dynamics of system-environment correlations for a dephasing qubit [23], we start from a typical model [29, 30] with total Hamiltonian

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{int}} + H_{\text{env}}, \quad (1)$$

by coupling a qubit non-dissipatively to a bath of harmonic oscillators through the choices [31–35]

$$\begin{aligned} H_{\text{sys}} &= \frac{\hbar\Omega}{2}\sigma_z \\ H_{\text{int}} &= \sigma_z \otimes \sum_{\lambda=1}^N \hbar g_{\lambda} a_{\lambda}^{\dagger} + \text{h.c.} \\ H_{\text{env}} &= \sum_{\lambda=1}^N \hbar \omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda}. \end{aligned} \quad (2)$$

Here  $\Omega$  denotes the energy difference between the qubit states and the coefficients  $g_{\lambda}$  describe the coupling strengths between the qubit and each environmental mode of frequency  $\omega_{\lambda}$  and annihilation and creation operators  $a_{\lambda}, a_{\lambda}^{\dagger}$ .

As environmental initial state we assume for each oscillator a thermal state  $\rho_{\text{therm}}^{\lambda} = (\bar{n}_{\lambda} + 1) \exp[-\hbar\omega_{\lambda} a_{\lambda}^{\dagger} a_{\lambda}/k_B T]$  with the mean thermal occupation number  $\bar{n}_{\lambda} = (\exp(\hbar\omega_{\lambda}/k_B T) - 1)^{-1}$  at temperature  $T$ . Initially, we assume no system-environment correlations such that the total initial state is simply given by the product  $\rho_{\text{tot}}(0) = \rho_{\text{sys}} \otimes \rho_{\text{therm}}$ . Accordingly, for the reduced system state  $\rho_{\text{red}}(t) = \text{Tr}_{\text{env}}[\rho_{\text{tot}}(t)]$  the dynamical map  $\mathcal{E}(t, 0): \rho_{\text{red}}(0) \rightarrow \rho_{\text{red}}(t)$  is completely positive with  $\rho_{\text{red}}(0) = \rho_{\text{sys}}$ .

Already at this stage we emphasise that on the reduced level, this dynamics can equally well be described by random unitary dynamics as will be elaborated upon in Sec. III.

The above quantum dephasing model (1) may be solved without any approximation. A possible approach to the time-local master equation is provided by the non-Markovian state diffusion approach to open systems [36, 37]. We find for the reduced density operator  $\rho_{\text{red}}(t) = \text{Tr}_{\text{env}}[\rho_{\text{tot}}(t)]$

$$\dot{\rho}_{\text{red}} = -i\frac{\Omega}{2}[\sigma_z, \rho_{\text{red}}] - \frac{\gamma(t)}{2}(\rho_{\text{red}} - \sigma_z \rho_{\text{red}} \sigma_z). \quad (3)$$

This equation is solved by

$$\rho_{\text{red}}(t) = \begin{pmatrix} \rho_{00} & \mathcal{D}(t)\rho_{01} \\ \mathcal{D}^*(t)\rho_{10} & \rho_{11} \end{pmatrix}, \quad (4)$$

with

$$\mathcal{D}(t) = \exp\left[-i\Omega t - \int_0^t \gamma(s) ds\right], \quad (5)$$

and where the  $\rho_{ij}$  represent the initial state of the qubit.

Equations (3) and (4) involve the time dependent dephasing rate  $\gamma(t)$  which by means of the spectral density of the environment  $J(\omega) = \sum_{\lambda=0}^N |g_{\lambda}|^2 \delta(\omega - \omega_{\lambda})$  can be written as

$$\gamma(t) = 4 \int_0^t ds \int_0^{\infty} d\omega J(\omega) \coth[\hbar\omega/2k_B T] \cos[\omega s]. \quad (6)$$

Later we will concentrate on environments that lead to periods in time with a *negative* dephasing rate.

Recently, we investigated system-environment correlations of this model [23] and found the useful representation

$$\rho_{\text{tot}}(t) = \int \frac{d^2\xi}{\pi} \frac{1}{\bar{n}} e^{-|\xi|^2/\bar{n}} \hat{P}(t; \xi, \xi^*) \otimes |\xi\rangle\langle\xi| \quad (7)$$

of the total state. It represents a partial P-representation where the environmental degrees of freedom are expanded in terms of coherent states  $|\xi\rangle$ . Here,  $\xi = (\xi_1, \xi_2, \dots)$  is a vector of complex numbers and we consistently make use of the notation  $d^2\xi/\pi := d^2\xi_1/\pi d^2\xi_2/\pi \dots$  (see also [38]). Furthermore, we symbolically write  $\exp[-|\xi|^2/\bar{n}]/\bar{n} := \prod_{\lambda} \exp[-|\xi_{\lambda}|^2/\bar{n}_{\lambda}]/\bar{n}_{\lambda}$  involving the mean thermal occupation number  $\bar{n}_{\lambda}$  of the  $\lambda$ -th environmental mode. The system part of the total state is encoded in a matrix-valued partial P-function  $\hat{P}(t)$  with values in the  $2 \times 2$  dimensional state space of the qubit.

In order to represent a solution of the total Schrödinger-von-Neumann equation with initial  $\rho_{\text{tot}}(0) = \rho_{\text{sys}}(0) \otimes \rho_{\text{therm}}$ , the partial P-function in (7) reads

$$\hat{P}(t; \xi, \xi^*) = \begin{pmatrix} \mathcal{A}^+(t; \xi, \xi^*)\rho_{00} & \mathcal{B}(t; \xi, \xi^*)\rho_{01} \\ \mathcal{B}^*(t; \xi, \xi^*)\rho_{10} & \mathcal{A}^-(t; \xi, \xi^*)\rho_{11} \end{pmatrix}. \quad (8)$$

Here,  $\mathcal{A}^{\pm} = \exp[-A(t) \pm \{(a(t)|\xi) + (\xi|a(t))\}]$  and  $\mathcal{B} = \exp[-i\Omega t] \exp[B(t) - \{(b(t)|\xi) - (\xi|b(t))\}]$ , where we have introduced the complex time dependent vectors  $a(t) = (a_1(t), a_2(t), \dots)$  and  $b(t)$  with scalar product  $(a(t)|\xi) \equiv \sum_{\lambda} a_{\lambda}^*(t)\xi_{\lambda}$  and vector components

$$a_{\lambda}(t) = \frac{1}{\bar{n}_{\lambda}} \int_0^t (g_{\lambda} e^{i\omega_{\lambda}s}) ds \quad (9)$$

$$b_{\lambda}(t) = \frac{2\bar{n}_{\lambda} + 1}{\bar{n}_{\lambda}} \int_0^t (g_{\lambda} e^{i\omega_{\lambda}s}) ds. \quad (10)$$

Furthermore, we use the abbreviations

$$\begin{aligned} A(t) &= 2 \text{Re} \int_0^t ds \int_0^s d\tau \left[ \sum_{\lambda} \frac{1}{\bar{n}_{\lambda}} |g_{\lambda}|^2 e^{-i\omega_{\lambda}(t-s)} \right] \\ B(t) &= 2 \text{Re} \int_0^t ds \int_0^s d\tau \left[ \sum_{\lambda} \frac{2\bar{n}_{\lambda} + 1}{\bar{n}_{\lambda}} |g_{\lambda}|^2 e^{-i\omega_{\lambda}(t-s)} \right]. \end{aligned}$$

Initially,  $\hat{P} = \rho_{\text{sys}}(0) = \rho_{\text{red}}(0)$  and note that there are no approximations necessary to achieve the result (8) and thus via (7) to obtain the exact state of the composite system (see also [23]).

### III. RANDOM UNITARY MODELS OF QUBIT DECOHERENCE

With an eye on experimental conditions, decoherence of qubits is often modelled by random unitary dynamics [27, 39]. In terms of a dynamical map, this implies that there exists a relation

$$\rho_{\text{red}}(t) = \sum_k p_k U_k \rho_{\text{red}}(0) U_k^\dagger \quad (11)$$

with suitably chosen probabilities  $p_k > 0$  and unitary maps  $U_k$ . Indeed, on the level of the reduced state, single-qubit decoherence can always be modelled in this way [24–26]. It is also worth noting that random unitary dynamics emerging from an open quantum system with environmental initial pure state can always be “undone” (quantum error correction) [40, 41].

The most straightforward random unitary realization of single-qubit decoherence with state (4) at time  $t$  is provided by the simple quantum operation

$$\begin{aligned} \rho_{\text{red}}(t) = & \left( \frac{1 + |\mathcal{D}(t)|}{2} \right) e^{-i\frac{\Omega}{2}t\sigma_z} \rho_{\text{red}}(0) e^{i\frac{\Omega}{2}t\sigma_z} \\ & + \left( \frac{1 - |\mathcal{D}(t)|}{2} \right) e^{-i\frac{\Omega}{2}t\sigma_z} \sigma_z \rho_{\text{red}}(0) \sigma_z e^{i\frac{\Omega}{2}t\sigma_z} \end{aligned} \quad (12)$$

which is obviously of the form (11) employing just two unitaries  $U_1 = \exp\{-i\Omega t\sigma_z/2\}$ ,  $U_2 = \exp\{-i\Omega t\sigma_z/2\}\sigma_z$  and probabilities  $p_{1,2} = (1 \pm |\mathcal{D}(t)|)/2$ . Recall that according to (5),  $|\mathcal{D}(t)| = \exp\{-\int_0^t \gamma(s)ds\}$ .

It is worth noting that the very same formal relation holds true for any two-time map  $\mathcal{E}(t, t') : \rho_{\text{red}}(t') \rightarrow \rho_{\text{red}}(t)$  such that

$$\begin{aligned} \rho_{\text{red}}(t) = & \left( \frac{1 + |\mathcal{D}(t, t')|}{2} \right) U_1(t - t') \rho_{\text{red}}(t') U_1^\dagger(t - t') \\ & + \left( \frac{1 - |\mathcal{D}(t, t')|}{2} \right) U_2(t - t') \rho_{\text{red}}(t') U_2^\dagger(t - t') \end{aligned} \quad (13)$$

with  $|\mathcal{D}(t, t')| = \exp\{-\int_{t'}^t \gamma(s)ds\}$ . As  $\gamma(s)$  need not be positive for all times (see later), the prefactor of the second contribution,  $\frac{1 - |\mathcal{D}(t, t')|}{2}$ , may turn negative for  $t'$  and  $t$  near times of negative  $\gamma(s)$ . Thus, for such times  $(t', t)$ , the map  $\mathcal{E}(t, t')$  in the form (13) ceases to take the form of a random unitary map. Indeed, using the Jamiolkowski isomorphism [42] it is straightforward to see that  $|\mathcal{D}(t, t')| < 1$  or

$$\int_{t'}^t \gamma(s)ds > 0 \quad (14)$$

is a necessary and sufficient condition for the dephasing map  $\mathcal{E}(t, t')$  defined above to be completely positive [17].

The random unitary form (12) of the dephasing map  $\mathcal{E}(t, 0)$  is simple but has the drawback of not representing an intuitive dynamical picture of the process due to the time dependence of the probabilities  $p_1 = p_1(t)$  and  $p_2 = p_2(t)$ . Here, we want to develop an alternative random unitary representation in a more systematic way that opens the door for further generalizations as explained below.

Recall that pure dephasing of a system in the basis  $\{|n\rangle\}$  implies a controlled-unitary form  $U_{\text{tot}}(t) = e^{-iH_{\text{tot}}t/\hbar} = \sum_n |n\rangle\langle n| \otimes U_n(t)$  of the total propagator [43]. Here,  $U_n(t) = e^{-iH_n t/\hbar}$  with  $H_n = \langle n|H_{\text{tot}}|n\rangle$  are system-state-dependent propagators for the environment (see also [44]). Pure dephasing is then given by the dynamics  $\langle n|\rho_{\text{red}}(t)|m\rangle = \text{tr}_{\text{env}}(U_n(t)\rho_{\text{env}}(0)U_m^\dagger(t)) \langle n|\rho_{\text{red}}(0)|m\rangle$ . In our case of a single qubit there is a single decoherence factor  $\mathcal{D}(t) = \text{tr}_{\text{env}}(U_0(t)\rho_{\text{env}}(0)U_1^\dagger(t))$  as in (5). The two propagators are determined by the environment Hamiltonians  $H_i = \langle i|H_{\text{tot}}|i\rangle$  with  $H_1 = \hbar\Omega/2 + \sum_{\lambda=1}^N \hbar g_\lambda (a_\lambda^\dagger + a_\lambda) + \sum_{\lambda=1}^N \hbar \omega_\lambda a_\lambda^\dagger a_\lambda$  and two sign changes for  $H_0$ . We next employ the Wigner representation of the environmental initial state  $W_0(\alpha, \alpha^*) = \int \frac{d^2\xi}{\pi} e^{\xi^* \alpha - \xi \alpha^*} \text{tr}(e^{\xi a^\dagger - \xi^* a} \rho_{\text{env}})$  and for the operator  $U_1^\dagger(t)U_0(t)$ , accordingly. The latter's Wigner Weyl symbol we denote by  $U_{01}(\alpha, \alpha^*) = \int \frac{d^2\xi}{\pi} e^{\xi^* \alpha - \xi \alpha^*} \text{tr}(e^{\xi a^\dagger - \xi^* a} U_1^\dagger(t)U_0(t))$ . We find

$$\mathcal{D}(t) = \int \frac{d^2\alpha}{\pi} W_0(\alpha, \alpha^*) U_{01}(\alpha, \alpha^*, t). \quad (15)$$

Due to the harmonic properties of the environment, the corresponding propagators  $U_n(t)$  are known explicitly and lead to the phase factor  $U_{01}(\alpha, \alpha^*, t) = \exp\{-i\Phi(\alpha, \alpha^*, t)\}$  with the phase

$$\Phi(\alpha, \alpha^*, t) = \frac{\Omega}{2}t - 2 \sum_\lambda g_\lambda \alpha_\lambda \int_0^t e^{-i\omega_\lambda s} ds + \text{c.c.} \quad (16)$$

We see that  $\mathcal{D}(t)$  is just an average over a random complex number of unit norm. For a thermal initial state the initial Wigner function  $W_0 = \frac{1}{\bar{n} + \frac{1}{2}} \exp\{-|\alpha|^2/(\bar{n} + \frac{1}{2})\}$  is positive. Thus, expression (15) corresponds to the random unitary representation for  $\mathcal{E}(t, 0)$ :

$$\rho_{\text{red}}(t) = \int \frac{d^2\alpha}{\pi} W_0(\alpha, \alpha^*) U_\alpha(t) \rho_{\text{red}}(0) U_\alpha^\dagger(t). \quad (17)$$

This corresponds to a random evolution  $U_\alpha = \exp\{-i \int_0^t H_\alpha(s)ds/\hbar\}$  of the qubit with diagonal stochastic Hamiltonian  $H_\alpha(t) = \sigma_z \left( \frac{\hbar\Omega}{2} - \sum_\lambda g_\lambda (\alpha_\lambda \int_0^t e^{-i\omega_\lambda s} ds + \alpha_\lambda^* \int_0^t e^{i\omega_\lambda s} ds) \right)$ . Note that in this representation the probability of occurrence of a particular unitary evolution is given by the value of the initial Wigner distribution and is thus time

independent. In this sense, the second random unitary representation (17) reflects an ensemble of experiments where the system dynamics is influenced by some (classical) stochastic process. Note that both random unitary representations of the dynamical map  $\mathcal{E}(t, 0)$  are exact – no restriction on the sign of  $\gamma(s)$  is necessary.

It may appear tempting to *define* a two-time map  $\mathcal{F}(t, t')$  through (17) with  $U_\alpha \rightarrow U_\alpha(t, t') = \exp\{-i \int_{t'}^t H_\alpha(s) ds / \hbar\}$ . However, it is clear that  $\mathcal{F}(t, t') \neq \mathcal{E}(t, t')$  unless  $t' = 0$ .

We close this section pointing out an interesting additional observation: the random unitary representation (17) for the quantum dephasing model is *not* restricted to single-qubit-dephasing. In fact, for an arbitrary system Hilbert space dimension, the very same construction works for all dephasing factors  $\mathcal{D}_{nm}(t) = \text{tr}_{\text{env}}(U_n(t) \rho_{\text{env}}(0) U_m^\dagger)$  of a quantum oscillator environment model. So even for larger Hilbert space dimension than two – on a local level – pure dephasing based on a quantum oscillator model like (1) *cannot* be distinguished from random unitary dynamics. For genuine quantum decoherence, one needs “more quantum mechanical” environments [26].

#### IV. MODEL OF DECOHERENCE WITH PERIODS OF NEGATIVE DEPHASING RATE

The physics of the harmonic oscillator environment model is encoded in its spectral density  $J(\omega)$ . As we are here interested in instances of negative dephasing rate, we chose a particular superomeric spectral density with sharp cutoff at frequency  $\omega_c$

$$J(\omega) = \kappa \frac{\omega^3}{\omega_c^2} \Theta(\omega - \omega_c) \quad (18)$$

and, for simplicity, concentrate on the high-temperature limit  $k_B T \gg \hbar \omega_c$ . In (18),  $\kappa$  denotes a dimensionless coupling constant. In the high-temperature limit the time dependent dephasing rate (6) can easily be evaluated analytically, we get

$$\hbar \gamma(t) = 8 \kappa k_B T \left( \frac{\sin(\omega_c t)}{(\omega_c t)^2} - \frac{\cos(\omega_c t)}{(\omega_c t)} \right). \quad (19)$$

As can be seen in Fig. 1,  $\gamma(t)$  turns negative in certain restricted periods, while the integral

$$\int_0^t \gamma(s) ds = \frac{8 \kappa k_B T}{\hbar \omega_c} \left( 1 - \frac{\sin(\omega_c t)}{(\omega_c t)} \right) \quad (20)$$

stays positive, as expected for a completely positive map.

Though many of our results do not rely on any special choice of  $J(\omega)$ , in the following, whenever we show figures, we will use the spectral density (18) and account for the high-temperature limit by choosing  $T = 10 \hbar \omega_c / k_B$ . Furthermore, we choose  $\kappa = 10^{-2}$  throughout this paper.

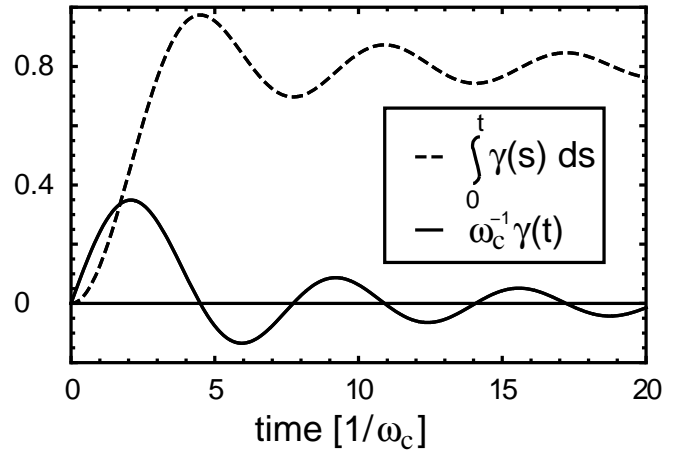


Figure 1: While the dephasing rate  $\gamma(t)$  clearly shows domains of negative values, the integrated quantity  $\int_0^t \gamma(s) ds$  stays positive, ensuring complete positivity of the dynamics.

#### V. TOTAL STATE AND PURITIES

Coupling to an environment leads to the built-up of correlations between system and environment and thus to changes in local entropies. From the point of view of information theory, regarding the reduced dynamics as a “channel” for quantum information, several quantities related to entropy are of interest [45]. Unfortunately, since all these quantities (quantum mutual information etc.) are based on von Neumann entropy  $-\text{tr}(\rho \log \rho)$ , they are hard to compute, unless one deals with very small systems or Gaussian states.

In a very first step we here want to determine purity  $P = \text{tr}(\rho^2)$  as an indicator for the mixedness of states, which is related to the linear entropy via  $S_L = 1 - P$ . Clearly, as with entropy, total purity  $P_{\text{tot}}$  is preserved under unitary evolution with  $H_{\text{tot}}$ . The sum of local purities  $P_{\text{sys}}(t)$ ,  $P_{\text{env}}(t)$  however, will be smaller as  $t > 0$ . For the initial product state we have  $P_{\text{tot}} = P_{\text{sys}} P_{\text{env}}$  and it appears natural for all  $t \geq 0$  to consider the difference of logarithms of  $P$  as a simple measure of “information correlation”

$$\begin{aligned} C(t) &= \log(P_{\text{tot}}) - \log(P_{\text{sys}}(t)) - \log(P_{\text{env}}(t)) \\ &= \log\left(\frac{P_{\text{sys}}(0)}{P_{\text{sys}}(t)}\right) + \log\left(\frac{P_{\text{env}}(0)}{P_{\text{env}}(t)}\right) \\ &\equiv C_{\text{sys}}(t) + C_{\text{env}}(t). \end{aligned} \quad (21)$$

Here the contributions  $C_{\text{sys}}$  and  $C_{\text{env}}$  correspond to the amount of correlations created between system and environment due to the increase of the local entropies in the two subsystems.

Having the total state (7) at hand, all these quantities can be determined easily for our dephasing qubit. For instance, the qubit purity is readily determined to give

$$P_{\text{sys}}(t) = \frac{1}{2} (1 + z^2 + (x^2 + y^2) |\mathcal{D}(t)|^2). \quad (22)$$

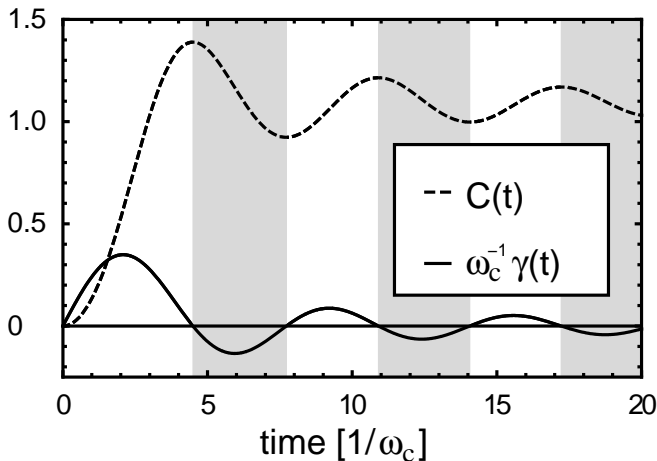


Figure 2: Dephasing rate  $\gamma(t)$  (solid) and information correlation  $C(t)$  (dashed) against time for a qubit with initial purity  $r = 0.98$ . While  $C(t)$  shows a monotonic increase for positive dephasing rates ( $\gamma(t) > 0$ ), we observe a decrease of the information correlation for  $\gamma(t) < 0$  (highlighted domains).

Here and in the following we denote by  $\mathbf{r} = (x, y, z) = \text{tr}(\sigma\rho)$  the coordinates of the Bloch vector of the *initial state of the qubit*. Somewhat more involved, yet still easy to determine is the purity of the environment. We find

$$P_{\text{env}}(t) = \frac{1}{2} (1 + z^2 + (1 - z^2)|\mathcal{G}(t)|^2) P_{\text{env}}(0) \quad (23)$$

with the initial environmental purity

$$\log P_{\text{env}}(0) = \int_0^\infty d\omega J(\omega) \log(\tanh(\hbar\omega/k_B T)). \quad (24)$$

In (22), the time dependence arises from the decoherence factor  $|\mathcal{D}(t)| = \exp(-\int_0^t \gamma(s) ds)$  of qubit dephasing with the rate  $\gamma(t) = 4 \int_0^t ds \int_0^\infty d\omega J(\omega) \coth[\hbar\omega/2k_B T] \cos[\omega s]$  from (6). By contrast, for the environment the time dependence is governed by a factor  $|\mathcal{G}(t)| = \exp(-\int_0^t \Gamma(s) ds)$  with a dual rate  $\Gamma(t) = 4 \int_0^t ds \int_0^\infty d\omega J(\omega) \tanh[\hbar\omega/2k_B T] \cos[\omega s]$ .

The rate of change of the correlation  $C(t)$  from (21) stems from the two contributions  $\dot{C} = \dot{C}_{\text{sys}} + \dot{C}_{\text{env}}$  with

$$\dot{C}_{\text{sys}}(t) = \frac{2\gamma(t)}{a|\mathcal{D}(t)|^2 + 1} \quad (25)$$

and

$$\dot{C}_{\text{env}}(t) = \frac{2\Gamma(t)}{b|\mathcal{G}(t)|^2 + 1} \quad (26)$$

where the initial state of the qubit determines the factors  $a = (1 + z^2)/(1 - z^2)$  and  $b = (1 + z^2)/(x^2 + y^2)$ .

In the high temperature limit, by means of (25) and (26) all quantities in (21) can be obtained readily. Reflecting the huge dimension of the environments

Hilbert space, it turns out that the environmental contribution is very small compared to  $C_{\text{sys}}(t)$ , and hence, we have  $C(t) \approx C_{\text{sys}}(t)$ . Therefore, from (25) one would expect  $C(t)$  to be correlated with  $\gamma(t)$ , independently of the special choice of  $J(\omega)$ . In Fig. 2 we compare the information correlation to the dephasing rate. Clearly, a change in  $C$  depends on the sign of  $\gamma$ , and thus  $C(t)$  decreases in the domains of negative  $\gamma(t)$ .

## VI. QUANTUM AND CLASSICAL SYSTEM-ENVIRONMENT CORRELATIONS

Having access to the total state we can also investigate the nature of system-environment correlations. In earlier work we have shown that quantum correlations need not exist in such open system models, in particular in the high-temperature limit. In such a case, the total state may still be separable and thus does not rely on any exchange of quantum information. Here we argue very much as in [23].

With the time and temperature dependent function

$$S(T, t) = 4 \int_0^t ds \int_0^s d\tau \int_0^\infty d\omega \times J(\omega) \exp[\hbar\omega/kT] \cos[\omega(s - \tau)] \quad (27)$$

we have shown that the total state is separable, as long as

$$S(t) \leq \ln \sqrt{\frac{1 - z^2}{x^2 + y^2}}. \quad (28)$$

In the high temperature limit, with our special choice of  $J(\omega)$  this quantity can be easily evaluated yielding

$$S(T, t) = 4\kappa \left( \frac{1}{2} - \frac{\sin[\omega_c t]}{\omega_c t} - \frac{\cos[\omega_c t]}{\omega_c t^2} + \frac{1}{\omega_c t^2} \right). \quad (29)$$

On the other hand, the total state turns into an entangled state at the latest when its partial transpose  $\rho_{\text{tot}}^{\text{PT}}$  yields a negative expectation value  $\langle \Psi | \rho_{\text{tot}}^{\text{PT}} | \Psi \rangle$  in some state  $|\Psi\rangle$  of the composite system [46]. By means of the representation (7) we have shown that with the time and temperature dependent function

$$E(T, t) = 8 \int_0^t ds \int_0^s d\tau \int_0^\infty d\omega \times J(\omega) \sinh[\hbar\omega/kT] \cos[\omega(s - \tau)] \quad (30)$$

entanglement is proven for

$$E(T, t) > \ln \left[ \frac{r - z^2}{x^2 + y^2} \right]. \quad (31)$$

Therefore  $E$  can be interpreted as “entanglement potential”, which implies quantum correlations if it exceeds the initial state dependent threshold given in (31). Again, in

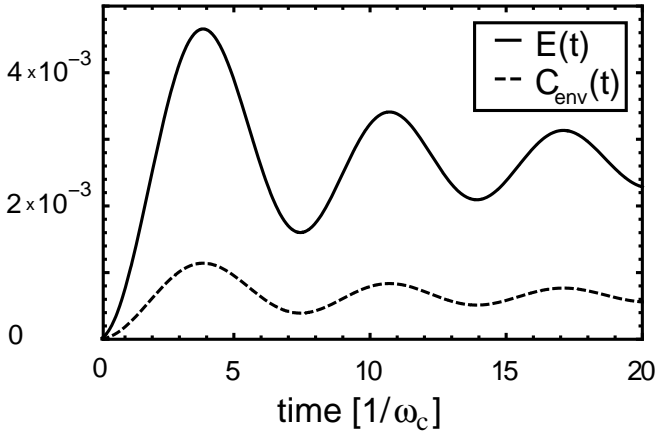


Figure 3:  $E(t)$  (solid) and  $C_{\text{env}}(t)$  (dashed) against time for a qubit with initial purity  $r = 0.98$ . Obviously, a change in the local entropy  $C_{\text{env}}$  is accompanied by a correlated change in the entanglement potential  $E$ . Note, that due to the condition (28) for separability of the total state, an increase of  $E$  does not lead to system-environment entanglement.

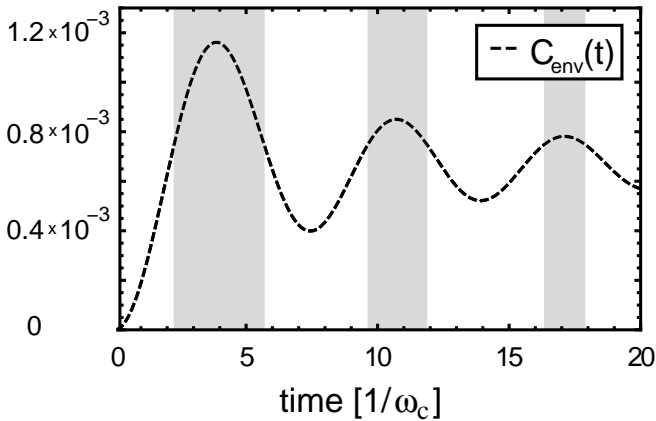


Figure 4:  $C_{\text{env}}(t)$  against time for a qubit with initial purity  $r = 0.997$ . The highlighted domains correspond to time intervals where the entanglement potential  $E > \ln[(r - z^2)/(x^2 + y^2)]$ . Clearly, an increase of the local entropy  $C_{\text{env}}(t)$  here leads to the growth of quantum correlations between system and environment.

the high temperature limit the special choice of  $J(\omega)$  enables us to obtain the following analytical expression

$$E(T, t) = \frac{8\kappa\hbar\omega_c}{k_B T} \left( \frac{1}{3} - 2 \frac{\cos[\omega_c t]}{(\omega_c t)^2} + 2 \frac{\sin[\omega_c t]}{(\omega_c t)^3} - \frac{\sin[\omega_c t]}{\omega_c t} \right). \quad (32)$$

We now use these expressions to relate the build up of correlations between system and environment to the environmental contribution  $C_{\text{env}}$  to the information correlation, introduced in the preceding section.

First of all, we want to stress that due to the high temperatures investigated here, the total state of system plus environment is separable at all times. This applies even if we choose an initial state in the equatorial plane of the Bloch sphere with purity of  $r = 0.98$ , as can be seen from (28). Nevertheless, it becomes obvious from Fig. 3 that the loss of information in the environment, described by  $C_{\text{env}}$ , is correlated with the magnitude of the entanglement potential  $E$ . However, these correlations are not of quantum nature, as already anticipated. Only if we employ even higher purities ( $r = 0.997$ ) like in Fig. 4, condition (28) ceases to hold true and entanglement can be detected in certain domains of time.

Let us stress that the relation between the two considered correlation related quantities exists, again, independently of the special choice of  $J(\omega)$ : since in the limit  $k_B T \gg \hbar\omega_c$  the general expressions for the derivative  $\dot{E}(T, t)$  of the entanglement potential and the dual rate  $\Gamma(T, t)$  essentially coincide, also the changes in  $C_{\text{env}}$  and  $E$  are correlated.

## VII. CONCLUSIONS

We have investigated the non-Markovian dynamics of a decohering qubit and its environment. Since it can be modeled by means of random unitary evolution, we have stressed that on the level of reduced dynamics, there is no need for a quantum description.

Nevertheless, considering the full dynamics of system plus environment, we have investigated the measure  $C = C_{\text{sys}} + C_{\text{env}}$  for system-environment correlations that emerge from an increase of the local entropies of the two subsystems. We have found this quantity to be closely related to the dephasing rate  $\gamma(t)$ , reflecting the non-Markovian character of the dynamics.

Furthermore, we have found that the environmental contribution  $C_{\text{env}}$  to the “information correlation”  $C$  is related to the “entanglement potential”  $E$  investigated previously in [23]. Though in most cases the total state of system and environment turns out to be separable, we have also shown that, even at high temperatures, quantum correlations can occur in the total state if the initial state of the qubit is sufficiently pure.

We are confident that our approach will be helpful for further investigations with respect to information flow in open system dynamics.

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